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THE DYNAMICS OF THE COLLISION BETWEEN A RIGID BODY AND A FLEXIBLE STRING AND MEMBRANE*

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An exact analysis of the boundary conditions at the point where an element of an absolutely flexible string or membrane meets the surface of a rigid body colliding with it as the "supersonic" velocity of the rigid body, the formulation of the problem concerning such a collision, accompanied by the tearing of the string of the rupture of the membrane, and the construction of its solution for the selfsimilar impact mode with constant velocity are given.

The principles of the mathematical theory for the collision of a solid with flexible structures in the form of strings or membranes were laid by Rakhmatullin /1/. A number of interesting results were obtained, but certain questions of the theory have not been clarified with finality. In particular, no final deductions were made regarding the set of possible formulations of the boundary conditions at the point where the element of flexible construction meets the surface of the solid. There was also no formulation of the problem of a collision accompanied by rupture of the flexible structure. The solution of these two questions is given below.

1. We will limit ourselves to examining the case when the material of the flexible structure is described by a linear law of elasticity in terms of conditional stresses while the collisions are such that the point of encounter of the structure element and the body surface is displaced at "supersonic" velocity over the structure, i.e., at a velocity exceeding the velocity of elastic wave propagation. Since abrupt bending of the structure (Fig.1) occurs at the point of encounter, i.e., a "jump" change in the momentum vector of the structure element as well as of its state of stress and strain is observed, a local reaction of the impacting body surface will be developed at this point which is modeled by a concentrated force. Taking the above into account for an idealized consideration of the problem, when the flexible structure (string or membrane) is considered as a one- or two-dimensional deformable continuum, the mechanics of the events in a small neighbourhood of the "break" point of the structure is modelled by introducing a "wave of strong discontinuity", i.e., a scheme with a jump-like change in the mechanical parameters at this point is introduced.

The Lagrange and Euler coordinates r and u, measured, respectively, along the strucutre from the point of its first contact with the impacting body and along the surface of this body, are introduced as the mechanical characteristics of the process in the one-dimensional case, as are the radial and azimuthal stresses $\bar{\sigma}_r, \bar{\sigma}_\theta$ (in the case of a string, one stress is along the filament $\bar{\sigma}_r = \bar{\sigma}_x$), corresponding to the strains

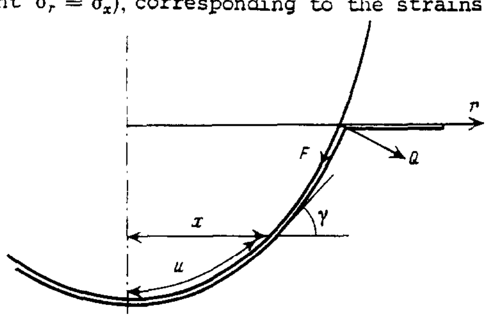

Fig.1

*Prikl.Matem.Mekhan.,49,1,85-93,1985

$$\varepsilon_r = \frac{\partial u}{\partial r} - 1, \quad \varepsilon_\theta = \frac{x(u)}{r} - 1 \tag{1.1}$$

where $x(u)$ is the Euler coordinate measured along the undeformed structure (Fig.1), the "conditional" stresses σ_r, σ_θ are determined from the formulas (the material density is assumed constant)

$$\sigma_r = \frac{\bar{\sigma}_r}{1+\varepsilon_r}, \quad \sigma_\theta = \frac{\bar{\sigma}_\theta}{1+\varepsilon_\theta} \tag{1.2}$$

and the law of elasticity is taken into account in the form

$$\sigma_r = \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta), \quad \sigma_\theta = \frac{E}{1-\nu^2}(\varepsilon_\theta + \nu\varepsilon_r) \tag{1.3}$$

where E, ν are the elastic modulus and Poisson's ratio.

In the case of a string we have $\sigma_x = E\varepsilon_x$ instead of (1.3).

The equations of motion are written in the form

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta \cos\gamma}{r} + \rho\frac{dV}{dt}\sin\gamma - \mu\left(\frac{\sigma_r}{r}\sin\gamma + \sigma_r\frac{d\gamma(u)}{du} + \rho\frac{dV}{dt}\cos\gamma\right)\operatorname{sign}\frac{\partial u}{\partial t} \tag{1.4}$$

in the case of the membrane and

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma_x}{\partial x} + \rho\frac{\partial V}{\partial t}\sin\gamma - \mu\left(\sigma_x\frac{d\gamma(u)}{du} + \rho\frac{dV}{dt}\cos\gamma\right)\operatorname{sign}\frac{\partial u}{\partial t} \tag{1.5}$$

in the case of a string. Here $V = V(t)$ is the velocity of the impacting body, γ is the angle between the unperturbed string (membrane) location and the tangent to the generatrix of the axisymmetric impacting body surface at a given point, the dependence $\gamma = \gamma(u)$ is determined by the body shape, ρ is the material of the deformable structure, and μ is the coefficient of friction of the structure on the body surface. Equations (1.4) and (1.5) are written in a coordinate system coupled to the impacting body, and consequently, inertial forces $\rho dV/dt$ occurred in them.

The relationships on the wave of strong discontinuity have the form

$$\begin{aligned} &\frac{b - v_1}{1+\varepsilon_{r1}} = \frac{b/\cos\gamma - v_2}{1+\varepsilon_{r2}} \\ &\rho(b - v_1)(v_2 - v_1\cos\gamma - V\sin\gamma) = (\sigma_{r1}\cos\gamma - \sigma_{r2} - \\ &\quad F)(1+\varepsilon_{r1}) \\ &\rho(b - v_1)(V\cos\gamma - v_1\sin\gamma) = (Q + \sigma_{r1}\sin\gamma)(1+\varepsilon_{r1}) \\ &v_1 = \left(\frac{\partial u}{\partial t}\right)_1, \quad v_2 = \left(\frac{\partial u}{\partial t}\right)_2, \quad b = V\operatorname{ctg}\gamma \end{aligned} \tag{1.6}$$

The subscripts 1, 2 refer to the quantities in front of and behind the front of the strong discontinuity, and F and Q are the tangential and normal components of the concentrated force, the reaction of the body surface, relative to the body surface.

An important assumption that an element of the structure (string, membrane) is located on the surface of the body directly after collision with this surface is made in writing relationships (1.6). In a certain sense this means that the collision is inelastic and can be considered as schematically taking into account the irreversible effects accompanying the local bending in a finite angle of the thin structure at the point where it collides with the body surface.

As already mentioned, we shall later consider the "supersonic" case when $b = V\operatorname{ctg}\gamma > a$, where

$$a = \sqrt{\frac{E}{(1-\nu^2)\rho}} \tag{1.7}$$

is the velocity of elastic wave propagation in the structure, and therefore, perturbations from the domain behind the front of the strong discontinuity cannot influence the parameters directly on this front, which should therefore be found independently of the behaviour of the solution in the domain behind the front. We will then have $\varepsilon_{r1} = 0$, $v_1 = 0$ and the relationships (1.6) will reduce to the form

$$v_2 = (1/\cos\gamma - 1 - \varepsilon_{r2})V\operatorname{ctg}\gamma \tag{1.8}$$

$$F = -\sigma_{r2} + \rho V \operatorname{ctg} \gamma \, (V \sin \gamma - v_2) \tag{1.9}$$

$$Q = \rho V^2 \operatorname{ctg} \gamma \cos \gamma \tag{1.10}$$

The condition for continuity of the displacements

$$x(u_2) = x(u_1) = r \tag{1.11}$$

should certainly be appended to the above relationships.

Using (1.11) and (1.1), we obtain $\varepsilon_{\theta 2} = 0$, and we have from (1.3) and (1.7)

$$\sigma_{r2} = \rho a^2 \varepsilon_{r2}, \; \sigma_{\theta 2} = \nu \sigma_{r2} \tag{1.12}$$

Substituting this expression for σ_{r2} into (1.9) we obtain

$$F = -\rho a^2 \varepsilon_{r2} + \rho V \operatorname{ctg} \gamma \, (V \sin \gamma - v_2) \tag{1.13}$$

which, together with (1.8) and (1.10) forms a system of three equations to determine the four quantities $F, Q, \varepsilon_{r2}, v_2$, i.e., the system is not closed.

To close it, one independent relationship must still be added. We can take the following conditions /2/ as such a relationship

$$v_2 = 0 \;\; (|F| < \mu_* Q) \tag{1.14}$$

which can be conserved only when the inequality written in parentheses is satisfied, where μ_* is the coefficient of Coulomb friction for the structure material-impacting body surface pair in terms of the surface reaction components F, Q. In general, μ_* can differ from μ.

It may be thought that conditions (1.14) will be conserved only for a certain set of values of the problem parameters whose boundary is governed by the condition

$$|F| = \mu_* Q \tag{1.15}$$

It is natural to assume that (1.15) must be taken as the closing condition outside the limits of the set; certainly $v_2 \neq 0$ already here so that the closing condition is written in the form

$$F = \mu_* Q \operatorname{sign} v_2 \tag{1.16}$$

However, as will be shown below, this does not exhaust all the possibilities. It turns out that for certain values of the problem parameters the quantities $\sigma_{r2}, \varepsilon_{r2}$ found by using the closing condition (1.16) become negative, which is not acceptable from physical considerations because flexible filaments or membranes cannot resist compression.

To construct a solution with physical meaning under these conditions, a scheme is necessary that takes account of the zero resistivity of the structure to compression. Evidently

$$\sigma_{r2} = 0 \tag{1.17}$$

will be the condition limiting the set of values of problem parameters for which the closure (1.16) will be suitable.

As such a scheme we take the requirement that relationship (1.17) must be satisfied for the set of problem parameter values for which a solution with the condition (1.16) will yield $\sigma_{r2} < 0$. The elasticity relationships (1.3) are already unacceptable here and condition (1.16) in addition to (1.17) is conserved.

Therefore, the set of all possible values of the problem parameters consists of three parts for which the closing relationship is (1.14), (1.16), and (1.17), respectively, whereupon (1.16) is also satisfied simultaneously. It will be shown below that within the limits of each of these subsets the solution is unique and continuous on their boundaries.

We first consider the closing condition (1.14). Substituting it into (1.8), (1.9) and (1.12), we obtain

$$\varepsilon_{r2} = 1/\cos \gamma - 1, \; F = \rho V^2 \cos \gamma - \rho a^2 (1/\cos \gamma - 1) \tag{1.18}$$

$$(\rho V^2 \cos \gamma - \rho a^2 (1/\cos \gamma - 1) < \mu_* \rho V^2 \operatorname{ctg} \gamma \cos \gamma)$$

It is interesting to note that the strain ε_{r2} and stress $\sigma_{r2}, \sigma_{\theta 2}$ (see (1.12)) are independent of the impact velocity V in the regime under consideration, and depend only on γ.

The boundary of the set of problem parameter values for which the solution (1.18) is applicable is determined by the transformation of the inequality in parentheses in (1.18) into the equality

$$\frac{\operatorname{ctg} \gamma \cos^2 \gamma \, (\operatorname{tg} \gamma - \operatorname{tg} \gamma_*)}{1 - \cos \gamma} = \frac{1}{M^2}; \quad M = \frac{V}{a}, \quad \operatorname{tg} \gamma_* = \mu_* \tag{1.19}$$

Relation (1.19) is shown in Fig.2 in the variables γ, M. Also shown there is the curve

$$M = \operatorname{tg} \gamma \tag{1.20}$$

yielding the lower bound for M that corresponds to a "supersonic" impact.

Thus, the solution is given by (1.18) in the domain above (1.20) and to the left of (1.19). To the right of (1.19) the solution should be constructed by using the closing condition (1.16), which yields

$$\varepsilon_{r2} = [M^2/(M^2 - \mathrm{tg}^2\,\gamma)]\,[1/\cos\gamma - 1 - \sin\gamma\,(\mathrm{tg}\,\gamma - \mathrm{tg}\,\gamma_*)] \qquad (1.21)$$
$$v_2 = V\,\mathrm{ctg}\,\gamma\,(1/\cos\gamma - 1 - \varepsilon_{r2})$$

It can be verified that ε_{r2} and v_2 determined by (1.18) and (1.21), agree on the line (1.19), i.e., the solution remains continuous on going over from one regime to the other. In the domain to the right of the boundary (1.19) there should be $v_2 > 0$ and $\varepsilon_{r2} > 0$, which indeed also holds.

The passage to the third possible regime is made on a line governed by the condition $\varepsilon_{r2} = 0$, where ε_{r2} is taken from the first formula in (1.21). This yields

$$\gamma = 2\gamma_* \qquad (1.22)$$

i.e., the boundary of the passage to the third regime is the line (1.22) in the γ, M plane. It can be verified that this line, the boundary (1.19), and the line (1.20) intersect at one point.

In the domain above the line (1.20) and to the right of (1.22), the solution is determined by the third regime characterized by condition (1.17). This regime corresponds to the collision process for which the string (membrane) element is wrinkled in the radial direction beyond the point of collision, is folded into a "bellows" so that its initial length is "shortened", i.e., $\varepsilon_{r2} < 0$, but this occurs for zero stresses. Under these conditions, as has been noted, it is already impossible to use the connection between σ_{r2} and ε_{r2} given by the elasticity relationships (1.3), and (1.16) must be used in conjunction with (1.17). Taking account of the above, the solution for this regime is given by the formulas

$$\varepsilon_{r2} = 1/\cos\gamma - 1 - \sin\gamma\,(\mathrm{tg}\,\gamma - \mathrm{tg}\,\gamma_*) \qquad (1.23)$$
$$v_2 = V\cos\gamma\,(\mathrm{tg}\,\gamma - \mathrm{tg}\,\gamma_*)$$

It can be seen that $v_2 > 0$, $\varepsilon_{r2} < 0$ is obtained by these formulas for $\gamma > 2\gamma_*$, i.e., wrinkling actually occurs. Comparing (1.23) and (1.21), the continuity of the change in ε_{r2}, v_2 during passage through the boundary (1.22) can be established.

Therefore, the complete solution of the problem has been constructed in the neighbourhood of the point of collision between the flexible string or membrane and the rigid body surface for a supersonic collision regime. This solution is unique, single-valued, and varies continuously during passage through the boundary of the domains of variation of the input parameters of the problem which are characterized by different closure conditions.

2. To describe the process of flexible deformable construction rupture under impact, it is useful to draw up a scheme of the process. In the case of the impact of a rigid body on a string, the criterion of no rupture can evidently be taken simply in the form

$$\max\sigma_x < \sigma_* \qquad (2.1)$$

where σ_* is the breaking strength of the string material. If the condition

$$\max\sigma_x = \sigma_* \qquad (2.2)$$

is achieved at some point of the string for a certain combination of the parameters, then for a set of problem parameters outside the boundary (2.2), the solution is constructed by introducing tearing of the string, i.e., $\sigma_x = 0$, at the site and at the time where and when condition (2.2) is achieved depending on the solution of the problem without rupture.

The situation is more complex when modeling the membrane rupture process. It is conceivable that the limiting condition will, as before, have the form (2.1), where σ_n must just be understood for σ_x, i.e., any normal stress. A more complex limit condition, containing different stress tensor invariants, can certainly also be formulated. The main difficulty, however, is in taking the correct rupture scheme which will not always be the local tearing in the case of a membrane, as in the case of a string, but can turn out to be a process of propagating discontinuities along the membrane.

Under static conditions, membrane rupture ordinarily occurs by propagation of one discontinuity. Under the conditions of high-velocity impact by a solid on a membrane, the rupture scheme will be different, at the site where the limit condition of the type (2.2) will first be achieved, several discontinuities will be generated and start to be propagated, and their number will generally increase with the impact velocity.

This assumption is based on the fact that at high velocity of the process the effects of unloading the membrane elements abutting on the tip of the discontinuity being propagated will not succeed in being propagated a noticeable distance and will unload the zone approaching the ultimate stress state; consequently, the material in such zones will be loaded "independently" will achieve the limit state and be ruptured, which will indeed appear in the form of the instantaneous generation and propagation of discontinuities. Such a process is observed in the dynamic rupture of three-dimensional solid brittle bodies subjected to intense dynamic action, for instance, the action of an explosion. The rupture failure process here is the propagation of a rupture front on which a large number of spall cracks or normal separation acts and the mathematical model with such a front is in good agreement with the real rupture process /3/. These considerations enable the process of dynamic membrane rupture under impact by a body to be modeled also by the insertion into the problem of a rupture front being propagated at whose leading side (in the unruptured part) a limit condition of the type (2.2) is achieved, while on its back side (in the ruptured part) the appropriate (rupturing) stress vanishes. Test shows that such a rupture scheme is actually observed, where the number of discontinuities grows with the impact velocity on the membrane /4/.

Below we borrow the construction of the solution of the simplest problem on impact by a wedge at constant velocity on an elastic string taking the possibility of rupture taken into account. Since the parameter σ_* is a constant on the dimensionality of the stress in condition (2.2), the problem under consideration will be selfsimilar, equation (1.4), resulting in the form

$$\rho \frac{\partial^2 u}{\partial t^2} = \rho c^2 \frac{\partial^2 u}{\partial x^2}; \quad \rho c^2 = E \tag{2.3}$$

has the solution

$$u = ctF(\xi), \ \xi = x/(ct) \tag{2.4}$$

where

$$F = A\xi + B; \ A, B = \text{const} \tag{2.5}$$

and the problem is to construct the solution of the initial impact problem by using (2.4) and (2.5), the conditions for achieving the limit state and rupture, and also the results of an analysis of the events at the point of encounter of the string element and the wedge surface, examined in Sect.1.

We will examine only the supersonic collision mode.

We start with the case when the solution beyond the break point is given by (1.18). According to (2.4) and (2.5), in this case $\varepsilon_x = A - 1, \ v = 0$ so that the relationships

$$A = 1/\cos\gamma, \ B = 0, \ 0 \leqslant x < Vt \operatorname{ctg}\gamma \tag{2.6}$$

yield the complete solution of the problem with the condition at the wedge apex

$$u|_{x=0} = 0 \tag{2.7}$$

For this case the limit condition has the form

$$\rho c^2 (1/\cos\gamma_p - 1) = \sigma_* \tag{2.8}$$

which is an equation in γ_p with the solution

$$\gamma_p = \gamma_p(\sigma_*/(\rho c^2)) \tag{2.9}$$

so that for $\gamma < \gamma_p$ there will be no rupture (tearing) of the string, while the string will be ruptured for $\gamma > \gamma_p$.

It is interesting that the limit condition does not contain the impact velocity and rupture will or will not depend on only the collision geometry (on the value of γ).

The solution with rupture will differ from the solution without rupture only in the domain $0 \leqslant x < ct$. Because of the condition $\sigma_x = 0$ for $x = 0$, which expresses tearing, we shall have $A = 1$ from (2.5) for F in this domain, while the condition of continuity of the displacements for $x = ct$ will yield $B = 1/\cos\gamma - 1$. In this domain the filament is stress-free and moves at the constant velocity $B \cdot c$.

Construction of the solution when the state beyond the break point is determined by (1.21) is somewhat more complex because, unlike the preceding, these formulas cannot describe the solution up to the wedge apex since there should be $v = 0$ there while $v \neq 0$ according to (1.21). These formulas are true just in the supersonic domain $ct < x < Vt \operatorname{ctg}\gamma$. In the domain $0 \leqslant x < ct$ formula (2.5) for $B = 0$ should be used, and the quantity A should be found from the continuity condition for u for $x = ct$, which yields

$$A = 1 + \varepsilon_{r2} + v_2/c \tag{2.10}$$

where ε_{r2}, v_2 are given by (1.21).

Therefore, part of the string is slowed down in the domain $0 \leqslant x < ct$, while the stress is $\rho v_2 c$ greater than in the domain beyond the break point. It is consequently conceivable that precisely this stress should be substituted as $\max\sigma_x$ in the limit condition, and this condition will become

$$\varepsilon_{r2} + v_2/c = \sigma_*/(\rho c^2) \tag{2.11}$$

In addition to γ this relationship also contains M, i.e., the limit condition on the γ, M plane will be a certain curve to the right of (1.19) and the left of (1.22). In the domain below this curve the motion will not be accompanied by rupture, while rupture will occur above it.

The solution will rupture will differ from the solution without rupture just in the domain $0 \leqslant x < ct$. Because of the boundary condition $\sigma_x = 0$ for $x = 0$, which expresses the fact of the occurence of string tearing at its constact point with the wedge apex (it is there that $\max\sigma_x$ first occurs), we obtain $A = 1$ for (2.5) while B is found from the condition of continuity of the displacements for $x = ct$

$$B = \varepsilon_{r2} + v_2/c \tag{2.12}$$

where ε_{r2}, v_2 are given by (1.21).

Therefore, for filament rupture a part of it $0 \leqslant x < ct$ is unloaded from the stresses and moves as a whole with velocity

$$v_p = v_2 + c\varepsilon_{r2} \tag{2.13}$$

We finally consider the solution of the problem for the values $\gamma > 2\gamma_*$, i.e., in a regime when wrinkling occurs behind the break point. In this case the interface between the domain with the wrinkling and the domain adjointing the wedge apex in which $\sigma_x > 0$, i.e., where the string is again stretched, will not be described by the equation of the "sonic" front $x = ct$, but will be determined because of selfsimilarity, by the equation

$$x = wt \tag{2.14}$$

where w is an unknown quantity to be determined during the solution of the problem. The ordinary conditions on a jump, relating the solutions (2.5) and (1.23), should be satisfied on this front of wrinkled filament expansion. The A and w are determined from these conditions ($B = 0$ in (2.5)) because of condition (2.7))

$$A = 1 + mM\cos\gamma\,(\mathrm{tg}\,\gamma - \mathrm{tg}\,\gamma_*),\ w/c = m = \sqrt{1+\xi^2} - \xi \tag{2.15}$$

$$\xi = \frac{\sin\gamma\,(\mathrm{tg}\,\gamma - \mathrm{tg}\,\gamma_*) + 1 - 1/\cos\gamma}{2M\cos\gamma\,(\mathrm{tg}\,\gamma - \mathrm{tg}\,\gamma_*)} > 0$$

In the expansion domain $0 \leqslant x < wt$ we have

$$v = 0,\ \sigma_x = \rho c^2 (A - 1) > 0 \tag{2.16}$$

The limit condition

$$A(\gamma, M, \gamma_*) - 1 = \sigma_*/(\rho c^2) \tag{2.17}$$

determines a curve in the domain $\gamma > 2\gamma_*$ below which the motion will not be accompanied by rupture, while above it will. It can be shown that this curve links up continuously with the analogous curve (2.11) in the domain $\gamma < 2\gamma_*$ and emerges on the "sonic" boundary (1.20) for finite M, i.e., for a certain $\gamma < \pi/2$.

For this regime the solution with rupture is quite simple, it will always agree with the solution in the bellows domain beyond the filament break point, i.e., in this case (1.23) is true everywhere in the domain $0 \leqslant x < Vt\,\mathrm{ctg}\,\gamma$.

The solutions with rupture constructed above are simultaneously solutions of the problem of a "supersonic" collision with constant velocity between a semi-infinite free string and a flat wall inclined at an angle γ to the string.

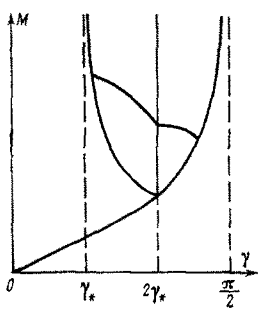

Fig.2

In Fig.2 we show domains with the different motion regimes and the line of limit states (2.11), (2.17). Note that if this line is disposed as shown in Fig.2, rupture is impossible in the domain to the left of the line (1.19). This is associated with the fact that the stresses behind the break point vary continuously on the line (1.19). But the stress to the left of the line (1.19) is constant everywhere in the solution of the problem of impact by a wedge, i.e., is equal to the stress behind the break point while the stress in the domain adjoining the wedge apex is greater than behind the break point in the solution of the problem of impact by a wedge for the domain to the right of the line (1.19). But this means that $\max\sigma_x$ to the left of (1.19) is less than $\max\sigma_x$ to the right of (1.19). However, according to Fig.2, the inequality $\max\sigma_x < \sigma_*$ is conserved to the right of (1.19), meaning that it will also be conserved to the left of (1.19). It also follows from this that rupture occurs to the left of (1.19) only when the curves (2.11) and (2.17) drop and shrink to the "triple" point $\gamma = 2\gamma_*$, $M = \mathrm{tg}\, 2\gamma_*$. The line (2.9) which will shift to the left as σ_* decreases will be the boundary separating the solution with rupture from the solution without rupture.

Thus, for very large values of σ_* rupture is possible only in the second and third regimes for high impact velocities, the curve of the limit states is in the domain of large values of M. As σ_* decreases, this curve drops montonically, and for a certain σ_* shrinks into a triple point. As σ_* decreases further, it is transformed into the segment of a line (2.9) which tends to the axis M as $\sigma_* \to 0$.

The solution of corresponding problems on the impact of a cone on a membrane can be constructed by exactly analogous methods by using the singularities of the solution at the break point of the structure and the scheme taken for the rupture process.

The results obtained here can be utilized in the general case of non-selfsimilar problems with curvilinear outlines of the impacting body surface.

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ON THE THEORY OF LONG WAVES IN AN INCLINED CHANNEL*

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A method resembling the asymptotic small-parameter method /1–3/, is used to study the long steady waves in an inclined channel, with the waves degenerating into solitons as their length tends to infinity. By analogy with the theory of stability of elastic rods, the process of transition from one-dimensional steady flow to two-dimensional flow, can be represented as instantaneous, with the result that all rectilinear stream lines becomes curved, but the values of the Froude and Reynolds numbers remain the same. It is shown that solutions of this type can exist, provided that the velocity of wave propagation and the value of the Reynolds number are nearly critical. Simple formulas are obtained for the wave profile, and the dependence of the wave propagation on the amplitude. If the Reynolds number is small and the angle of inclination of the channel is nearly $\pi/2$, the same formulas hold even without the assumption that the Reynolds number is nearly critical. The method opens up the possibility of proving existence and uniqueness theorems by analogy with /1–3/. Technical difficulties arise in connection with the estimates for Green's function for the biharmonic operator.

1. Formulation of the problem. Consider the two-dimensional steady flow of a homogeneous, incompressible heavy viscous fluid with a free boundary, over a rectilinear bottom inclined at an angle α to the horizontal. We shall assume that the two-dimensional flow is caused by instantaneous loss of stability of a one-dimensional flow characterized by the Reynolds number $R = Q/\nu$ and Froude number $F = gH^3/Q^2$ (Q is the flow rate and H is the depth of the stream. We shall write the equations of motion in a coordinate system moving in a direction parallel to the channel bottom with wave velocity c. The origin of coordinates is chosen at the free unperturbed boundary, and the y axis is parallel to the force of gravity.

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